# Some Fixed Point Theorems In Partially Ordered G-Metric Spaces And Applications To Global Existence And Attractivity Results For Nonlinear Functional Integral Equations

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***Abstract*** **- In this paper some results for the two weakly increasing mappings f and g with respect to partial ordering relation $\leq$ are proved in generalized metric spaces which are further applied to prove the existence results concerning the global attractivity and global asymptotic attractivity for a certain functional nonlinear integral equation. Our existence results include several existence as well as attractivity results obtained earlier by Banas and Dhage [8], Hu and Yan [19] and Banas and Rzepka [7] as special case.**

***Keywords*** **-Measure of noncompactness, Fixed point theorem, G-Metric spaces, Functional integral equation, Attractivity, Asymptotic attractivity, Existence theorem**

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## I. INTRODUCTION

It is well-known that the fixed point theorems which are obtained using the mixed arguments from different branches of mathematics are very rich in applications to allied areas of mathematics, particularly to the theory of nonlinear differential and integral equations. The existence of fixed points in partially ordered sets has been at the center of active research. In fact, the existence of fixed point in partially ordered sets has been investigated in [23].

Moreover, Ran and Reurings [23] applied their results to matrix equations. In [24], O'Regan and PetrusCel gave some existence results for the Fredholm and Volterra type. The notion of G-metric space was introduced by Mustafa and Sims [21] as a generalization of the notion of metric spaces. Many other authers also studied fixed point results in G-metric space see [1,12,25]. In fact the study of common fixed points of mappings satisfying certain contractive conditions has been at the center of strong research activity. The following definition is introduced by Mustafa and Sims [21].

**Definition 1.1 (see [21]).** Let X be a nonempty set and let $G : X \times X \times X \to R$ be a function satisfying the following properties:
$(G_1)$ $G(x, y, z) = 0$ if $x = y = z$,
$(G_2)$ $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$,
$(G_3)$ $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$,
$(G_4)$ $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$,symmetry in all three variables,
$(G_5)$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$.
Then the function G is called a generalized metric, or, more specifically, a G-metric on X, and the pair (X,G) is called a G-metric space.

**Definition 1.2 (see [21]).** Let (X,G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X, a point $x \in X$ is said to be the limit of the sequence $\{x_n\}$, if $lim_{n,m\to\infty} G(x, x_n, x_m) = 0$, and we say that the sequence $\{x_n\}$ is G-convergent to x or $\{x_n\}$ G-converges to x.
Thus, $x_n \to x$ in a G-metric space $(X, G)$ if for any $\epsilon > 0$, there exists $k \in N$ such that $G(x, x_n, x_m) < \epsilon$ for all $m, n \in k$.

**Proposition 1.3 (see [21]).** Let $(X, G)$ be a G-metric space. Then the following are equivalent:
(1) $\{x_n\}$ is G-convergent to x;
(2) $G(x_n, x_n, x) \to 0$ as $n \to \infty$;
(3) $G(x_n, x, x) \to 0$ as $n \to \infty$;
(4) $G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.

**Definition 1.4 (see [21]).** Let $(X, G)$ be a G-metric space, a sequence $\{x_n\}$ is called G-Cauchy if for every $\epsilon > 0$, there is $k \in N$ such that $G(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \in k$; that is $G(x_n, x_m, x_l) < 0$ as $n, m, l \to +\infty$.

**Proposition 1.5 (see [21]).** Let (X,G) be a G-metric space. Then the following are equivalent:
(1) the sequence $x_n$ is G-Cauchy;
(2) for every $\in > 0$, there is $k \in N$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \in k$.

**Definition 1.6 (see [21]).** Let (X,G) and $(X', G')$be G-metric spaces, and let $f : (X, G) \to (X', G')$ be a function. Then f is said to be G-continuous at a point $a \in X$ if and only if for every $\epsilon > 0$, there is $\delta > 0$ such that $x, y \in X$ and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \epsilon$. A function f is G-continuous at X if and only if it is G-continuous at all $a \in X$.

**Proposition 1.7 (see [21]).** Let (X,G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables. Every G-metric on X will define a metric $d_G$ on X by

$$d_G(x, y) \leq G(x, y, y) + G(y, x, x), \ \forall\ x, y \in X. \qquad (1.1)$$

For a symmetric G-metric space,

$$d_G(x,y) = 2G(x,y,y),\ \forall\ x,y \in X. \tag{1.2}$$

However, if G is not symmetric, then the following inequality holds:

$$\frac{3}{2}G(x,y,y) \le d_G(x,y) \le 3G(x,y,y),\ for\ all\ x,y \in X. \tag{1.3}$$

The following are examples of G-metric spaces.

**Example 1.8 (see [21]).** Let (R, d) be the usual metric space. Define $G_S$ by

$$G_s(x,y,z) \le d(x,y) + d(y,z) + d(x,z) \tag{1.4}$$

$\forall\, x,y,z \in R$. Then it is clear that $(R, G_s)$ is a G-metric space.

**Example 1.9 (see [21]).** Let $X = \{a,b\}$. Define G on $X \times X \times X$ by

$$G(a,a,a) = G(b,b,b) = 0,$$

$$G(a,a,b) = 1,\ G(a,b,b) = 2 \tag{1.5}$$

and extend G to $X \times X \times X$ by using the symmetry in the variables. Then it is clear that (X,G) is a G-metric space.

**Definition 1.10 (see [21]).** A G-metric space (X,G) is called G-complete if every G-Cauchy sequence in (X,G) is G-convergent in (X,G). The notion of weakly increasing mappings was introduced in by Altun and Simsek [5].

**Definition 1.11 (see [5]).** Let $(X, \le)$ be a partially ordered set. Two mappings $F, G : X \to X$ are said to be weakly increasing if $F_x \le G(F_x)$ and $G_x \le F(G_x)$, for all $x \in X$.
Two weakly increasing mappings need not be nondecreasing.

**Example 1.12 (see [5]).** Let $X = R$, endowed with the usual ordering. Let $F, G : X \to X$ defined by

$$\begin{aligned} F_x &= x, 0 \le x \le 1, \\ &= 0, 1 < x < +\infty, \end{aligned}$$

$$\begin{aligned} g_x &= \sqrt(X), 0 \le x \le 1 \\ &= 0, 1 < x < +\infty. \end{aligned} \tag{1.6}$$

Then F and G are weakly increasing mappings. Note that F and G are not nondecreasing.
The aim of this paper is to study a number of fixed point results for two weakly increasing mappings f and g with respect to partial ordering relation $\le$ in a generalized metric space.

## II. FIXED POINT THEOREMS

**Theorem 2.1.** Let $(X, \le)$ be a partially ordered set and suppose that there exists G-metric in X such that $(X, G)$ is G-complete. Let $f, g : X \to X$ be two weakly increasing mappings with respect to $\le$. Suppose there exist nonnegative real numbers $\alpha, \beta, \gamma$ and $\delta$ with $\alpha + 2\beta + 2\gamma + 2\delta < 1$ such that

$$\begin{aligned} G(fx,gy,gy) \le{}& \alpha G(x,y,y) + \beta[G(x,fx,fx) + G(y,gy,gy)] \\ &+ \gamma[G(x,gy,gy) + G(y,fx,fx)] \\ &+ \delta[G(x,fy,fy) + G(y,gx,gx)], \end{aligned} \tag{2.1}$$

$$\begin{aligned} G(gx,fy,fy) \le{}& \alpha G(x,y,y) + \beta[G(x,gx,gx) + G(y,fy,fy)] \\ &+ \gamma[G(x,fy,fy) + G(y,gx,gx)] \\ &+ \delta[G(x,gy,gy) + G(y,fx,fx)], \end{aligned} \tag{2.2}$$

for all comparative $x, y \in X$. If $f$ or $g$ is continuous, then $f$ and $g$ have a common fixed point $u \in X$.

**Proof.** By inequality (2.2), we have

$$\begin{aligned} G(gy,fx,fx) \le{}& \alpha G(y,x,x) + \beta[G(y,gy,gy) + G(x,fx,fx)] \\ &+ \gamma[G(y,fx,fx) + G(x,gy,gy)] \\ &+ \delta[G(y,gx,gx) + G(x,fy,fy)]. \end{aligned} \tag{2.3}$$

If X is a symmetric G-metric space, then by adding inequalities (2.1) and (2.3), we obtain

$$\begin{aligned} G(fx,gy,gy) + G(gy,fx,fx) \le{}& \alpha[G(x,y,y) + G(y,x,x)] \\ &+ 2\beta[G(x,fx,fx) + G(y,gy,gy)] \\ &+ 2\gamma[G(x,gy,gy) + G(y,fx,fx)] \\ &+ 2\delta[G(x,fy,fy) + G(y,gx,gx)] \end{aligned} \tag{2.4}$$

which further implies that

$$\begin{aligned} d_G(fx,fy) \le{}& \alpha d_G(x,y) + \beta[d_G(x,fx) + d_G(y,gy)] \\ &+ \gamma[d_G(x,gy) + d_G(y,fx)] + \delta[d_G(x,fy) + d_G(y,gx)] \end{aligned} \tag{2.5}$$

for all $x, y \in X$ with $0 \le \alpha + 2\beta + 2\gamma + 2\delta < 1$ and the fixed point of $f$ and $g$ follows from [2]. Now if X is not a symmetric G-metric space. Then by the definition of metric $(X, d_G)$ and inequalities (2.1) and (2.3), we obtain

$$\begin{aligned} & d_G(fx,gy) \\ & \le \alpha[G(x,y,y) + G(x,x,y)] + 2\beta[G(x,fx,fx) + G(y,gy,gy)] \\ & + 2\gamma[G(x,gy,gy) + G(y,fx,fx)] + 2\delta[G(x,fy,fy) + G(y,gx,gx)] \\ & \le \alpha d_G(x,y) + 2\beta[\frac{2}{3}d_G(x,fx) + \frac{2}{3}d_G(y,gy)] \\ & + 2\gamma[\frac{2}{3}d_G(x,gy) + \frac{2}{3}d_G(y,fx)] \\ & = \alpha d_G(x,y) + \frac{4}{3}\beta[d_G(x,fx) + d_G(y,gy)] \\ & + \frac{4}{3}\gamma[d_G(x,gy) + d_G(y,fx)] + \frac{4}{3}\delta[d_G(x,fy) + d_G(y,gx)], \\ & \text{for all } x \in X. \end{aligned} \tag{2.6}$$

Here, the contractivity factor $\alpha + (8/3)\beta + (8/3)\gamma + (8/3)\delta$ may not be less than 1.
Therefore metric gives no information. In this case, for given $x_0 \in X$, choose $x_1 \in X$ such that $x_1 = fx_0$. Again choose

$x_2 \in X$ such that $gx_1 = x_2$. Also, we choose $x_3 \in X$ such that $x_3 = fx_2$. Continuing as above process, we can construct a sequence $x_n$ in X such that $x_{2n+1} = fx_{2n},\ n \in N \cup \{0\}$ and $x_{2n+2} = gx_{2n+1},\ n \in N \cup \{0\}$. Since $f$ and $g$ are weakly increasing with respect to $\leq$, we have

$$\begin{aligned} x_1 = fx_0 &\leq g(fx_0) = gx_1 = x_2 \\ &\leq f(gx_1) = fx_2 = x_3 \leq g(fx_2) = gx_3 = x_4 \leq ... \end{aligned} \tag{2.7}$$

Thus from (2.1), we have

$$\begin{aligned} &G(x_{2n+1}, x_{2n+2}, x_{2n+2}) \\ &= G(fx_{2n}, gx_{2n+1}, gx_{2n+1}) \\ &\leq \alpha G(x_{2n}, x_{2n+1}, x_{2n+1}) \\ &+ \beta[G(x_{2n}, fx_{2n}, fx_{2n}) + G(x_{2n+1}, gx_{2n+1}, gx_{2n+1})] \\ &+ \gamma[G(x_{2n}, gx_{2n+1}, gx_{2n+1}) + G(x_{2n+1}, fx_{2n}, fx_{2n})] \\ &+ \delta[G(x_{2n}, fx_{2n+1}, fx_{2n+1}) + G(x_{2n+1}, gx_{2n}, gx_{2n})] \\ &= \alpha G(x_{2n}, x_{2n+1}, x_{2n+1}) \\ &+ \beta[G(x_{2n}, x_{2n+1}, x_{2n+1}) + G(x_{2n+1}, x_{2n+2}, x_{2n+2})] \\ &+ \gamma[G(x_{2n}, x_{2n+2}, gx_{2n+2}) + G(x_{2n+1}, x_{2n+1}, x_{2n+1})] \\ &+ \delta[G(x_{2n}, x_{2n+2}, x_{2n+2}) + G(x_{2n+1}, x_{2n+1}, x_{2n+1})] \\ &= (\alpha + \beta)G(x_{2n}, x_{2n+1}, x_{2n+1}) + \beta G(x_{2n+1}, x_{2n+2}, x_{2n+2}) \\ &+ \gamma G(x_{2n}, x_{2n+2}, x_{2n+2}) + \delta(G(x_{2n}, x_{2n+2}, x_{2n+2})). \end{aligned} \tag{2.8}$$

By $(G_5)$,

$$G(x_{2n+1}, x_{2n+2}, x_{2n+2}) = \frac{\alpha + \beta + \gamma + \delta}{1 - \beta - \gamma - \delta} G(x_{2n}, x_{2n+1}, x_{2n+1})$$

Also, we have

$$\begin{aligned} &G(x_{2n}, x_{2n+1}, x_{2n+1}) \\ &= G(gx_{2n-1}, fx_{2n}, fx_{2n}) \\ &= \alpha G(x_{2n-1}, x_{2n}, x_{2n}) \\ &+ \beta[G(x_{2n-1}, gx_{2n-1}, gx_{2n-1}) + G(x_{2n}, fx_{2n}, fx_{2n})] \\ &+ \gamma[G(x_{2n-1}, fx_{2n}, fx_{2n}) + G(x_{2n}, gx_{2n-1}, gx_{2n-1})] \\ &+ \delta[G(x_{2n-1}, gx_{2n}, gx_{2n}) + G(x_{2n}, fx_{2n-1}, fx_{2n-1})] \\ &= \alpha G(x_{2n-1}, x_{2n}, x_{2n}) \\ &+ \beta[G(x_{2n-1}, gx_{2n-1}, gx_{2n-1}) + G(x_{2n}, x_{2n+1}, x_{2n+1})] \\ &+ \gamma[G(x_{2n-1}, x_{2n+1}, x_{2n+1}) + G(x_{2n}, x_{2n}, x_{2n})] \\ &+ \delta[G(x_{2n-1}, x_{2n+1}, x_{2n+1}) + G(x_{2n}, x_{2n}, x_{2n})] \\ &= (\alpha + \beta)G(x_{2n-1}, x_{2n}, x_{2n}) + \beta G(x_{2n}, x_{2n+1}, x_{2n+1}) \\ &+ \gamma G(x_{2n-1}, x_{2n+1}, x_{2n+1}) \\ &+ \delta[G(x_{2n-1}, x_{2n}, x_{2n}) + G(x_{2n}, x_{2n+1}, x_{2n+1})] \end{aligned} \tag{2.9}$$

$$G(x_{2n}, x_{2n+1}, x_{2n+1}) \leq \frac{\alpha + \beta + \gamma + \delta}{1 - \beta - \gamma - \delta} G(x_{2n-1}, x_{2n}, x_{2n}) \tag{2.10}$$

Let

$$K = \frac{\alpha + \beta + \gamma + \delta}{1 - \beta - \gamma - \delta} \tag{2.11}$$

Then by (2.9) and (2.11), we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq KG(x_{n-1}, x_n, x_n) \tag{2.12}$$

Thus, if $x_0 = x_1$, we get $G(x_n, x_{n+1}, x_{n+1}) = 0$ for each $n \in N$. Hence $x_n = x_0$ for each $n \in N$. Therefore $\{x_n\}$ is G-Cauchy. So we may assume that $x_0 \neq x_1$. Let $n, m \in N$ with $m > n$. By axiom $(G_5)$ of the definition of G-metric space, we have

$$\begin{aligned} G(x_n, x_m, x_m) \leq\ & G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &+ ... + G(x_{m-1}, x_m, x_m) \end{aligned} \tag{2.13}$$

By (2.13), we get

$$\begin{aligned} G(x_n, x_m, x_m) &\leq K^n G(x_0, x_1, x_1) + K^{n+1} G(x_0, x_1, x_1) \\ &+ ... + K^{m-1} G(x_0, x_1, x_1) \\ &\leq \frac{k^n}{1-k} G(x_0, x_1, x_1) \end{aligned} \tag{2.14}$$

On taking limit $m, n \to \infty$, we have

$$lim_{m,n \to \infty} G(x_n, x_m, x_m) = 0 \tag{2.15}$$

So we conclude that $(x_n)$ is a Cauchy sequence in X. Since X is G-complete, then it yields that $(x_n)$ and hence any subsequence of $(x_n)$ converges to some $u \in X$. So that, the subsequences $(x_{2n+1}) = fx_{2n}$ and $(x_{2n+2}) = gx_{2n+1}$ converge to u. First suppose that f is G-continuous. Since $(x_{2n})$ converges to u, we get $(fx_{2n})$ converges $fu$. By the uniqueness of limit we get $fu = u$. Claim: $gu = u$. Since $u \leq u$, by inequality (2.1), we have

$$\begin{aligned} G(u, gu, gu) &\leq G(fu, gu, gu) \\ &\leq \alpha G(u, u, u) + \beta[G(u, fu, fu) + G(u, gu, gu)] \\ &+ \gamma[G(u, gu, gu) + G(u, fu, fu)] \\ &+ \delta[G(u, fu, fu) + G(u, gu, gu)] \\ &\leq (\beta + \gamma + \delta)G(u, gu, gu). \end{aligned} \tag{2.16}$$

Since $\beta + \gamma + \delta < 1$, we get $G(u, gu, gu) = 0$. Hence $gu = u$. If $g$ is G-continuous, by similar argument as above we show that $g$ and $f$ have a common fixed point.

**Theorem 2.2.** Let $(X, \leq)$ be a partially ordered set and suppose that there exists G-metric in X such that (X,G) is G-complete. Let $f, g : X \to X$ be two weakly increasing mappings with respect to $\leq$. Suppose there exist nonnegative real numbers $\alpha, \beta, \gamma$ and $\delta$ with $\alpha + 2\beta + 2\gamma + 2\delta < 1$ such that

$$\begin{aligned} G(fx, gy, gy) \leq\ & \alpha G(x, y, y) + \beta[G(x, fx, fx) + G(y, gy, gy)] \\ &+ \gamma[G(x, gy, gy) + G(y, fx, fx)] \\ &+ \delta[G(x, fy, fy) + G(y, gx, gx)] \end{aligned}$$

$$\begin{aligned} G(gx, fy, fy) \leq\ & \alpha G(x, y, y) + \beta[G(x, gx, gx) + G(y, fy, fy)] \\ &+ \gamma[G(x, fy, fy) + G(y, gx, gx)] \\ &+ \delta[G(x, gy, gy) + G(y, fx, fx)] \end{aligned}$$

(2.17)

for all comparative $x, y \in X$. Assume that X has the following property:
$(P)$ If $(x_n)$ is an increasing sequence converges to x in X, then $x_n \leq x$ for all $n \in N$. Then $f$ and $g$ have a common fixed point $u \in X$.

**Proof.** As in the proof of Theorem 2.1, we construct an increasing sequence $(x_n)$ in X such that $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$. Also, we can show $(x_n)$ is G-Cauchy. Since X is G-complete, there is $u \in X$ such that $(x_n)$ is converges to $u \in X$. Thus $x_{2n}, x_{2n+1}, fx_{2n}$ and $gx_{2n+1}$ converge to u. Since X satisfies property $(P)$, we get that $x_n \leq u$, for all $n \in N$. Thus $x_{2n}$ and u are comparative. Hence by inequality (2.1), we have

$$\begin{aligned} G(fx_{2n}, gu, gu) \leq \alpha G(x_{2n}, u, u) &\\ + \beta[G(x_{2n}, fx_{2n}, fx_{2n}) + G(u, gu, gu)] &\\ + \gamma[G(x_{2n}, gu, gu) + G(u, fx_{2n}, fx_{2n})] &\\ + \delta[G(x_{2n}, fu, fu) + G(u, gx_{2n}, gx_{2n})] & \end{aligned} \tag{2.18}$$

On letting $n \to \infty$, we get

$$G(u, gu, gu) \leq (\beta + \gamma + \delta) G(u, gu, gu). \tag{2.19}$$

Since $\beta + \gamma + \delta < 1$, we get $G(u, gu, gu) = 0$. Hence $gu = u$. By similar argument, we may show that $u = fu$.

**Corollary 2.3.** Let $(X, \leq)$ be a partially ordered set, and suppose that $(X, G)$ is a G-complete metric space. Let $f : X \to X$ be a continuous mapping such that $fx \leq f(fx)$, for all $x \in X$. Suppose there exist nonnegative real numbers $\alpha, \beta, \gamma$ and $\delta$ with $\alpha + 2\beta + 2\gamma + 2\delta < 1$ such that

$$\begin{aligned} G(fx, fy, fy) \leq \alpha G(x, y, y) + \beta[G(x, fx, fx) + G(y, fy, fy)] &\\ + \gamma[G(x, fy, fy) + G(y, fx, fx)] &\\ + \delta[G(x, fy, fy) + G(y, fx, fx)] & \end{aligned} \tag{2.20}$$

for all comparative $x, y \in X$. Then $f$ has a fixed point $u \in X$.

**Proof.** It follows from Theorem 2.1 by taking $g = f$.

**Corollary 2.4.** Let $(X, \leq)$ be a partially ordered set and suppose that there exists G-metric in X such that (X,G) is G-complete. Let $f : X \to X$ be a mapping such that $fx \leq f(fx)$ for all $x \in X$. Suppose there exist nonnegative real numbers $\alpha, \beta, \gamma$ and $\delta$ with $\alpha + 2\beta + 2\gamma + 2\delta < 1$ such that

$$\begin{aligned} G(fx, fy, fy) \leq \alpha G(x, y, y) + \beta[G(x, fx, fx) + G(y, fy, fy)] &\\ + \gamma[G(x, fy, fy) + G(y, fx, fx)] &\\ + \delta[G(x, fy, fy) + G(y, fx, fx)] & \end{aligned} \tag{2.21}$$

for all comparative $x, y \in X$. Assume that X has the following property:
(P) If $(x_n)$ is an increasing sequence converges to x in X, then $x_n \leq x$ for all $n \in N$. Then $f$ has fixed point $u \in X$.

**Proof.** It follows from Theorem 2.2 by taking $g = f$.

The fixed point theorems (2.1), (2.2), (2.3) and (2.4) are quite general and includes some fixed point theorems studied earlier by an author as special case. For example, if we take $\gamma = 0$ in (2.1), (2.2), (2.3) and (2.4) then these are reduced to fixed point theorems discussed in W. Shatanawi [26].

## III. FUNCTIONAL INTEGRAL EQUATIONS

Let $\mathbb{R}$ denote the real line and $\mathbb{R}_+$, the set of nonnegative real numbers. consider the following nonlinear functional integral equation (in short FIE)

$$x(t) = F\left(t, f(t, x(\alpha(t))), \int_0^{\beta(t)} g(t, s, x(\gamma(s)))\, ds\right) \tag{3.1}$$

for all $t \in \mathbb{R}_+$, where $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $F : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\alpha, \beta, \gamma : \mathbb{R}_+ \to \mathbb{R}_+$.

By a **solution** of the FIE (3.1) we mean a function $x \in C(\mathbb{R}_+, \mathbb{R})$ that satisfies the equation (3.1), where $C(\mathbb{R}_+, \mathbb{R})$ is the space of real-valued functions defined and continuous on $\mathbb{R}_+$.

The FIE (3.1) is of interest, since it is a quite general and includes some nonlinear integral equations studied earlier by various authors as special cases. For example, if we take $F(t, x, y) = x + y$ for all $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$, then it is reduced to a functional integral equation discussed in Banas and Dhage [8],

$$x(t) = f(t, x(\alpha(t))) + \int_0^{\beta(t)} g(t, s, x(\gamma(s)))\, ds. \tag{3.2}$$

Similarly, if $F(t, x, y) = x\,(q(t) + y)$ for all $t \in \mathbb{R}_+$, then the integral equation (3.2) reduces to the the integral equations studied in Dhage and O'Regan [18],

$$x(t) = \left[f(t, x(\alpha(t)))\right] \left(q(t) + \int_0^{\beta(t)} g(t, s, x(\gamma(s)))\, ds\right). \tag{3.3}$$

Again if, $F(t, x, y) = q(t) + y$, the FIE (3.1) reduces to the well-known Urysohn-Volterra functional integral equation

$$x(t) = q(t) + \int_0^{\beta(t)} g(t, s, x(\gamma(s)))\, ds. \tag{3.4}$$

Similarly, if $\alpha(t) = \beta(t) = \gamma(t) = t$, $f(t, x) = x$, then the FIE (3.1) reduces to the nonlinear integral equation studied in Hu and Yan [19],

$$x(t) = F\left(t, x(t), \int_0^t g(t, s, x(s))\, ds\right). \tag{3.5}$$

Finally, if $\alpha(t) = \beta(t) = \gamma(t) = t$, $F(t, x, y) = 1 + xy$, $f(t, x) = x$, and $g(t, s, y) = \dfrac{t}{t+s}\phi(s)\, x$, the FIE (3.1) reduces to the well-known Chandrasekhar's integral equation in radioative transfer,

$$x(t) = 1 + x(t) \int_0^1 \frac{t}{t+s}\phi(s)\, x(s)\, ds. \tag{3.6}$$

Thus, our FIE (3.1) is more general and so it is worthwhile for investigating different aspects of the solutions. Some special cases of the FIE (3.1) have also been discussed in Burton and Zhang [10],Burton and Furumochi [11] and Bedre [9] for existence and other characterizations such as stability of the solution on $\mathbb{R}_+$.

Let $E$ be a Banach space and let $\mathcal{P}_p(E)$ denote the class of all non-empty subsets of $E$ with property $p$. Here $p$ may be $p$ = closed (in short cl), $p$ = bounded (in short bd), $p$ = relatively compact (in short rcp) etc. Thus, $\mathcal{P}_{cl}(E), \mathcal{P}_{bd}(E), \mathcal{P}_{cl,bd}(E)$ and $\mathcal{P}_{rcp}(E)$ denote respectively the classes of closed, bounded, closed and bounded and relatively compact subsets of $E$. A function $d_H : \mathcal{P}_p(E) \times \mathcal{P}_p(E) \times \mathcal{P}_p(E) \to \mathbb{R}^+$ defined by

$$\begin{aligned} G_H(A,B,C) = \max&\left\{\sup_{a\in A} D(a,B)\ ,\ \sup_{b\in B} D(b,A)\right\} \\ &+ \max\left\{\sup_{a\in A} D(b,C)\ ,\ \sup_{b\in B} D(c,B)\right\} \\ &+ \max\left\{\sup_{a\in A} D(a,C)\ ,\ \sup_{b\in B} D(c,A)\right\} \end{aligned} \tag{3.7}$$

satisfies all the conditions of a metric on $\mathcal{P}_p(E)$ and is called a Hausdorff-Pompeiu G-metric on $E$, where $D(a,B) = \inf\{\|a-b\| : b \in B\}$. It is known that the hyperspace $(\mathcal{P}_{cl}(E), G_H)$ is a complete G-metric space.

The axiomatic way of defining the measures of noncompactness has been adopted in several papers in the literature. See Akhmerov *et al* [4], Appell [6], Deimling [13], Väth [22] and the references therein. In this paper, we also adopt the same axiomatic definition of the measure of noncompactness in a Banach space using G-Metric space. A few details along this line appears in Dhage [14]. We need the following definitions in the sequel.

**Definition 3.1.** A sequence $\{A_n\}$ of non-empty sets in $\mathcal{P}_p(E)$ is said to converge to a set $A$, called the **limiting set** if $G_H(A_n, A) \to 0$ as $n \to \infty$. A mapping $\mu : \mathcal{P}_p(E) \to \mathbb{R}^+$ is called continuous if for any sequence $\{A_n\}$ in $\mathcal{P}_p(E)$ we have that

$$G_H(A_n, A) \to 0 \Longrightarrow |\mu(A_n) - \mu(A)| \to 0 \quad \text{as} \quad n \to \infty.$$

**Definition 3.2.** A mapping $\mu : \mathcal{P}_p(E) \to \mathbb{R}^+$ is called nondecreasing if $A, B \in \mathcal{P}_p(E)$ are any two sets with $A \subseteq B$, then $\mu(A) \le \mu(B)$, where $\subseteq$ is a order relation by inclusion in $\mathcal{P}_p(E)$.

Now we are equipped with the necessary details to define the measures of noncompactness for a bounded subset of the Banach space $E$.

**Definition 3.3(Dhage [14]).** A function $\mu : \mathcal{P}_{bd}(E) \to \mathbb{R}^+$ is called a *measure of noncompactness* if it satisfies

$1^o$ $\emptyset \neq \mu^{-1}(0) \subset \mathcal{P}_{rcp}(E)$,

$2^o$ $\mu(\overline{A}) = \mu(A)$, where $\overline{A}$ denotes the closure of $A$,

$3^o$ $\mu(\mathrm{Conv}\, A) = \mu(A)$, where $\mathrm{Conv}\, A$ denotes the convex hull of $A$,

$4^o$ $\mu$ is nondecreasing, and

$5^o$ if $\{A_n\}$ is a decreasing sequence of sets in $\mathcal{P}_{bd}(E)$ such that $\lim\limits_{n\to\infty} \mu(A_n) = 0$, then the limiting set $A_\infty = \lim\limits_{n\to\infty} \overline{A}_n$ is non-empty.

The family $\ker \mu$ described in $1^o$ is said to be the *kernel of the measure of noncompactness* $\mu$ and

$$\ker \mu = \{A \in \mathcal{P}_{bd}(E) \mid \mu(A) = 0\} \subset \mathcal{P}_{rcp}(E).$$

Observe that the limiting set $A_\infty$ from $5^o$ is a member of the family $\ker \mu$. In fact, since $\mu(A_\infty) \le \mu(\overline{A}_n) = \mu(A_n)$ for any $n$, we infer that $\mu(A_\infty) = 0$. This yields that $A_\infty \in \ker \mu$. This simple observation will be essential in our further investigations.

Now, we give a useful definition.

**Definition 3.4.** A mapping $Q : E \to E$ is called $\mathcal{M}$-**set-Lipschitz** if there exists a continuous monotonic function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu(Q(A)) \le \phi(\mu(A))$ for all $A \in \mathcal{P}_{bd}(E)$ with $Q(A) \in \mathcal{P}_{bd}(E)$, where $\phi(0) = 0$. Sometimes we call the function $\phi$ to be a $\mathcal{M}$-**function** of $Q$ on $E$. In the special case, when $\phi(r) = kr, k > 0$, $Q$ is called a $k$-**set-Lipschitz** mapping and if $k < 1$, then $Q$ is called a $k$-**set-contraction** on $E$. Further, if $\phi(r) < r$ for $r > 0$, then $Q$ is called a **nonlinear** $\mathcal{M}$-**set-contraction** on $E$.

**Remark 3.1.** Let us denote by $\mathrm{Fix}(Q)$ the set of all fixed points of the operator $Q$ which belong to $X$. It can be shown that the set $\mathrm{Fix}(Q)$ existing in Corollary (2.4) belongs to the family $\ker \mu$. In fact if $\mathrm{Fix}(Q) \notin \ker \mu$, then $\mu(\mathrm{Fix}(Q)) > 0$ and $Q(\mathrm{Fix}(Q)) = \mathrm{Fix}(Q)$. Now from nonlinear $\mathcal{M}$-set-contraction it follows that $\mu(Q(\mathrm{Fix}(Q))) \le \phi(\mu(\mathrm{Fix}(Q)))$ which is a contradiction since $\phi(r) < r$ for $r > 0$. Hence $\mathrm{Fix}\,(Q) \in \ker \mu$.

Our further considerations will be placed in the Banach space $BC(\mathbb{R}_+, \mathbb{R})$ consisting of all real functions $x = x(t)$ defined, continuous and bounded on $\mathbb{R}_+$. This space is equipped with the standard supremum norm

$$||x|| = \sup\{|x(t)| :\ t \in \mathbb{R}_+\}\ .$$

For our purposes we will use the Hausdorff or ball measure of noncompactness in $BC(\mathbb{R}_+, \mathbb{R})$. A handy formula for Hausdorff measure of noncompactness useful in applications is defined as follows. Let us fix a nonempty and bounded subset $X$ of the space $BC(\mathbb{R}_+, \mathbb{R})$ and a positive number $T$. For $x \in X$ and $\varepsilon \ge 0$, denote by $\omega^T(x,\varepsilon)$ *the modulus of continuity* of the function $x$ on the closed and bounded interval $[0,T]$ defined by

$$\omega^T(x,\varepsilon) = 2\sup\{|x(t) - x(s)| :\ t, s \in [0,T],\ |t-s| \le \varepsilon\}\ .$$

Next, let us put

$$\omega^T(X,\varepsilon) = \sup\{\omega^T(x,\varepsilon) :\ x \in X\}\ ,$$

$$\omega_0^T(X) = \lim_{\varepsilon\to 0} \omega^T(X,\varepsilon)\ .$$

Finally, we define

$$\omega_0(X) = \lim_{T\to\infty} \omega_0^T(X) \ .$$

Now, for a fixed number $t \in \mathbb{R}_+$ let us denote

$$X(t) = \{x(t) : x \in X\}$$

and

$$\|X(t)\| = \sup\{|x(t)| \ : \ x \in X\}.$$

Finally, let us consider the functions $\mu_a$ defined on the family $\mathcal{P}_{bd}(X)$ by the formulas

$$\mu_a(X) = \max\left\{\omega_0(X)\,,\,\limsup_{t\to\infty} \operatorname{diam} X(t)\right\}. \tag{3.8}$$

and

$$\mu_b(X) = \max\left\{\omega_0(X)\,,\,\limsup_{t\to\infty}\ \|QX(t)\|\right\}. \tag{3.9}$$

It can be shown as in Dhage [17] that the functions $\mu_a$, and $\mu_b$ are measures of noncompactness in the space $BC(\mathbb{R}_+,\mathbb{R})$. The kernels ker $\mu_a$ and ker $\mu_b$ of the measures $\mu_a$ and $\mu_b$ consist of nonempty and bounded subsets $X$ of $BC(\mathbb{R}_+,\mathbb{R})$ such that functions from $X$ are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundles formed by functions from $X$ tends to zero at infinity.

The above expressed property of ker $\mu_a$ and ker $\mu_b$ permits us to characterize solutions of the integral equations considered in this paper.

In order to introduce further concepts used in this paper, let $\Omega$ be a subset of $BC(\mathbb{R}_+,\mathbb{R})$. Let $Q : BC(\mathbb{R}_+,\mathbb{R}) \to BC(\mathbb{R}_+,\mathbb{R})$ be a operator and consider the following operator equation in $E$,

$$Qx(t) = x(t) \tag{3.10}$$

for all $t \in \mathbb{R}_+$. Below we give different characterizations of the solutions for the operator equation (3.10) on $\mathbb{R}_+$.

**Definition 3.5.** We say that solutions of the equation (3.10) are **locally attractive** if there exists a closed ball $\overline{\mathcal{B}}_r(x_0)$ in the space $BC(\mathbb{R}_+,\mathbb{R})$ for some $x_0 \in BC(\mathbb{R}_+,\mathbb{R})$ such that for arbitrary solutions $x = x(t)$ and $y = y(t)$ of the equation (3.10) belonging to $\overline{\mathcal{B}}_r(x_0) \cap \Omega$ we have that

$$\lim_{t\to\infty} (x(t) - y(t)) = 0. \tag{3.11}$$

In the case when the limit (3.11) is uniform with respect to the set $B(x_0, r) \cap \Omega$, i.e., when for each $\varepsilon > 0$ there exists a $T > 0$ such that

$$|x(t) - y(t)| \le \varepsilon \tag{3.12}$$

for all $x, y \in \overline{\mathcal{B}}_r(x_0) \cap \Omega$ being solutions of (3.10) and for $t \ge T$, we will say that solutions of equation (3.10) are **uniformly locally attractive** on $\mathbb{R}_+$.

**Definition 3.5.** A solution $x = x(t)$ of equation (3.10) is said to be **globally attractive** if (3.11) holds for each solution $y = y(t)$ of (3.10) in $\Omega$. In other words, we may say that solutions of the equation (3.10) are globally attractive if for arbitrary solutions $x(t)$ and $y(t)$ of (3.10) in $\Omega$, the condition (3.11) is satisfied. In the case when the condition (3.11) is satisfied uniformly with respect to the set $\Omega$, i.e., if for every $\varepsilon > 0$ there exists a $T > 0$ such that the inequality (3.12) is satisfied for all $x, y \in \Omega$ being the solutions of (3.10) and for $t \ge T$, we will say that solutions of the equation (3.10) are **uniformly globally attractive** on $\mathbb{R}_+$.

The following definitions appear in Dhage [15].

**Definition 3.6.** A line $y(t) = c$, where $c$ a real number, is called a **attractor** for a solution $x \in BC(\mathbb{R}_+,\mathbb{R})$ of the equation (3.10) if $\lim_{t\to\infty}[x(t) - c] = 0$. In this case the solution $x$ of the equation (3.10) is also called to be asymptotic to the line $y(t) = c$ and the line is an asymptote for the solution $x$ on $\mathbb{R}_+$.

Now we introduce the following definition useful in the sequel.

**Definition 3.7.** We say that solutions of the equation (3.10) are **locally asymptotically attractive** if there exists a closed ball $\overline{\mathcal{B}}_r(x_0)$ in the space $BC(\mathbb{R}_+,\mathbb{R})$ for some $x_0 \in BC(\mathbb{R}_+,\mathbb{R})$ such that for arbitrary solutions $x = x(t)$ and $y = y(t)$ of equation (3.10) belonging to $\overline{\mathcal{B}}_r(x_0) \cap \Omega$ we have that the condition (3.11) is satisfied and there is a line which is a common attractor to them on $\mathbb{R}_+$. In the case when condition (3.11) is satisfied uniformly, that is, if for every $\varepsilon > 0$ there exists a $T > 0$ such that the inequality (3.12) is satisfied for $t \ge T$ and for all $x, y$ being the solutions of (3.10) and have a line as a common attractor, we will say that solutions of the equation (3.10) are **uniformly locally asymptotically attractive** on $\mathbb{R}_+$.

**Definition 3.8.** The solutions of the equation (3.10) are said to be **globally asymptotically attractive** if for any two solutions $x = x(t)$ and $y = y(t)$ of the equation (3.10), the condition (3.11) is satisfied and there is a line which is a common attractor to them on $\mathbb{R}_+$. In the case when condition (3.11) is satisfied uniformly, that is, if for every $\varepsilon > 0$ there exists a $T > 0$ such that the inequality (3.12) is satisfied for $t \ge T$ and for all $x, y$ being the solutions of (3.10) and have a line as a common attractor, we will say that solutions of the equation (3.10) are **uniformly globally asymptotically attractive** on $\mathbb{R}_+$.

**Remark 3.2.** Let us mention that the concept of global attractivity of solutions is recently introduced in Hu and Yan [19] while the concepts of local and global asymptotic attractivity have been presented in Dhage [15, 16]. Similarly, the concepts of uniform local and global attractivity (in the above sense) were introduced in Banas and Rzepka [7]. Note that global attractivity implies the local attractivity and global asymptotic attractivity implies the local asymptotic attractivity of solutions for the functional integral equations on unbounded intervals of real line. However, the converse of the above statements may not be true. In this sense the global attractivity and global asymptotic attractivity results are more general than the local attractivity and local asymptotic attractivity results for the FIE (3.1) on unbounded intervals of real line.

## IV. GLOBAL ATTRACTIVITY RESULTS

In this section we prove our main local attractivity results of this paper.

Let $X = BC(R^+, R)$ be the set of all continuous and bounded functions defined on $R^+$. Define

$$G : X \times X \times X \to R^+$$

by

$$G(x, y, z) = \sup |x(t)-y(t)| + \sup |x(t)-z(t)| + \sup |y(t)-z(t)|.$$

Then $(X, G)$ is a G-metric space. Define an ordered relation $\leq$ on $X$ by

$$x \preceq y \quad iff \quad x(t) \preceq y(t), \quad \forall t \in R^+$$

We consider the following hypotheses in the sequel.

($A_0$) The functions $\alpha, \beta, \gamma : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and $\alpha(t) \geq t$ and $\beta(t) \geq t$ for all $t \in \mathbb{R}_+$.

($A_1$) The function $F : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist functions $\ell_1, \ell_2 \in BC(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$|F(t, x_1, x_2) - F(t, y_1, y_2)| \leq \ell_1(t)|x_1 - y_1| + \ell_2(t)|x_2 - y_2|$$

for all $(t, x_1, x_2), (t, y_1, y_2) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$. Moreover, we assume that $L_1 = \sup_{t\geq 0} \ell_1(t)$.

($A_2$) The function $t \mapsto F(t, 0, 0)$ is bounded with $F_0 = \sup_{t\geq 0} |F(t, 0, 0)|$.

($A_3$) The function $f : \mathbb{R}_+ \times \mathbb{R} \to \times\mathbb{R}$ is continuous and there exists a $k_1 \in BC(\mathbb{R}_+, \mathbb{R}_+)$ and a real number $M > 0$ such that

$$|f(t, x_1) - f(t, y_1)| \leq \frac{k_1(t)|x_1 - y_1|}{M + |x_1 - y_1|}$$

for all $(t, x_1), (t, y_1) \in \mathbb{R}_+ \times \mathbb{R}$. Moreover, assume that $K_1 = \sup_{t\geq 0} k_1(t)$.

($A_4$) The function $t \mapsto f(t, 0)$ is bounded with $f_0 = \sup_{t\geq 0} |f(t, 0)|$.

($A_5$) There exists a continuous function $b : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|g(t, s, x)| \leq b(t, s)$$

for all $t, s \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Moreover, we assume that

$$\lim_{t\to\infty} \int_0^{\beta(t)} b(t, s)\, ds = 0.$$

($A_6$) for each $t \in R^+$, one has

$$\begin{aligned} &F(t, x(\alpha(t)), x(\gamma(s))) \\ &\leq F\left(t, f(t, x(\alpha(t))), \int_0^{\beta(t)} g(t, s, x(\gamma(s)))\, ds\right) \end{aligned}$$

**Remark 4.1.** The Lipschitz condition given for the nonlinearity $f$ in the hypothesis ($A_1$) is more general than the usual Lipschitz condition. In fact if $K_1 < M$, then it reduces to the Lipschitz condition of the function $f$,

$$|f(t, x_1) - F(t, y_1)| \leq k_1(t)|x_1 - y_1|.$$

**Remark 4.2.** Note that the functions $w : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$w(t) = sup \int_0^{\beta(t)} b(t, s)\, ds$$

is continuous and in view of hypothesis ($A_5$) which further implies that the number $W = \sup_{t\geq 0} w(t)$ is finite.

**Theorem 4.1.** Assume that the hypotheses ($A_0$)-($A_6$) hold. Further if $L_1 K_1 \leq M$, then the FIE (3.1) admits a solution. Moreover, solutions are uniformly globally attractive on $\mathbb{R}_+$.

**proof.** Set $E = BC(\mathbb{R}_+, \mathbb{R})$. Define a mapping $Q$ on $E$ by

$$Qx(t) = F\left(t, f(t, x(\alpha(t))), \int_0^{\beta(t)} g(t, s, x(\gamma(s)))\, ds\right) \quad (4.1)$$

for $t \in \mathbb{R}_+$.

First we show that $Q$ maps $E$ into itself. As all the functions on the right hand side of (4.1) are continuous, the function $(Qx)$ is continuous on $\mathbb{R}_+$ for each $x \in E$. Again, by hypotheses ($A_1$)-($A_2$), we obtain:

$$\begin{aligned} |Qx(t)| &\leq \left| F\Big(t, f(t, x(\alpha(t))), \int_0^{\beta(t)} g(t, s, x(\gamma(s)))\, ds, \Big) \right| \\ &\leq \left| F\Big(t, f(t, x(\alpha(t))), \int_0^{\beta(t)} g(t, s, x(\gamma(s)))\, ds\Big) - F(t, 0, 0) \right| \\ &\qquad + |F(t, 0, 0)| \\ &\leq \ell_1(t)\left[|f(t, x(\alpha(t))) - f(t, 0)| + |f(t, 0)|\right] \\ &+ \ell_2(t) \left| \int_0^{\beta(t)} g(t, s, x(\gamma(s)))|\, ds \right| + F_0 \\ &\leq \frac{\ell_1(t)k_1(t)|x(\alpha(t))|}{M + |x(\alpha(t))|} + \ell_1(t) f_0 \\ &\leq \frac{L_1 K_1 \|x\|}{M + \|x\|} + \ell_2(t) \int_0^{\beta(t)} b(t, s)\, ds \\ &+ F_0 + L_1 f_0 \\ &\leq L_1 K_1 + w(t) + F_0 + L_1 f_0 \\ &\leq L_1 K_1 + W + F_0 + L_1 f_0 \\ &\leq r \end{aligned} \quad (4.2)$$

for all $t \in \mathbb{R}_+$. This shows that $(Qx)$ is a bounded function on $\mathbb{R}_+$. As a result, $Q$ defines a mapping $Q : E \to E$. Next, we show that $Q$ satisfies all the conditions of corollary (2.4). Define a closed ball $\overline{\mathcal{B}}_r(0)$ in $E$ centered at origin of radius $r = L_1 K_1 + W + F_0 + L_1 f_0$. Then $Q$ defines a map $Q : E \to \overline{\mathcal{B}}_r(0)$ and particular, $Q : \overline{\mathcal{B}}_r(0) \to \overline{\mathcal{B}}_r(0)$. Because of this fact, solutions of the FIE (3.1) if exist, are global in nature.

$$\begin{aligned}
&Qx(t)\\
&= F\left(t, f(t, x(\alpha(t))), \int_0^{\beta(t)} g(t, s, x(\gamma(s)))\, ds\right)\\
&\leq F\,(t, f(t, f(t, x(\alpha(t))))\\
&\quad , \int_0^{\beta(t)} g\left(t, s, \int_0^{\beta(t)} g(t, s, x(\gamma(s)))\, ds\right)\, ds\Big)\\
&= F\left(t, f(t, Qx(\alpha(t))), \int_0^{\beta(t)} g(t, s, Qx(\gamma(s)))\, ds\right)\\
&= Q\,(Qx(t)) \qquad (4.3)
\end{aligned}$$

Thus we have $Qx(t) \leq Q(Qx(t))$, for all $x \in \overline{\mathcal{B}}_r(0)$.

Next, we show that $Q$ is a $k$-set-contraction on $\overline{\mathcal{B}}_r(0)$. Let $\varepsilon > 0$ be given and let $T > 0$ be a fixed real number. Choose a function $x \in \overline{\mathcal{B}}_r(0)$ and $t_1, t_2 \in [0, T]$ such that $|t_1 - t_2| \leq \varepsilon$. Then in view of our assumptions ($A_1$)-($A_3$),

$$\begin{aligned}
&G\,(Qx(t_1), Qx(t_2), Qx(t_2))\\
&= 2\ \sup\ |Qx(t_1) - Qx(t_2)|\\
&\leq 2\ \sup\ \Big|F\Big(t_1, f(t_1, x(\alpha(t_1))), \int_0^{\beta(t_1)} g(t_1, s, x(\gamma(s)))\, ds\Big)\\
&\qquad - F\Big(t_2, f(t_2, x(\alpha(t_2))), \int_0^{\beta(t_2)} g(t_2, s, x(\gamma(s)))\, ds\Big)\Big|\\
&\leq 2\ \sup\ \Big|F\Big(t_1, f(t_1, x(\alpha(t_1))), \int_0^{\beta(t_1)} g(t_1, s, x(\gamma(s)))\, ds\Big)\\
&\qquad - F\Big(t_1, f(t_2, x(\alpha(t_2))), \int_0^{\beta(t_2)} g(t_2, s, x(\gamma(s)))\, ds\Big)\Big|\\
&\quad + 2\ \sup\ \Big|F\Big(t_1, f(t_2, x(\alpha(t_2))), \int_0^{\beta(t_2)} g(t_2, s, x(\gamma(s)))\, ds\Big)\\
&\qquad - F\Big(t_2, f(t_2, x(\alpha(t_2))), \int_0^{\beta(t_2)} g(t_2, s, x(\gamma(s)))\, ds\Big)\Big|\\
&\leq 2\ \sup\ \ell_1(t)\big|f(t_1, x(\alpha(t_1))) - f(t_2, x(\alpha(t_2)))\big|\\
&\quad + 2\ \sup\\
&\qquad \ell_2(t)\left|\int_0^{\beta(t_1)} g(t_1, s, x(s))\, ds - \int_0^{\beta(t_2)} g(t_2, s, x(\gamma(s)))\, ds\right|\\
&\quad + \omega_r^T(F, \varepsilon)\\
&\leq 2\ \sup\\
&\qquad \ell_1(t)\big|f(t_1, x(\alpha(t_1))) - f(t_1, x(\alpha(t_2)))\big|\\
&\quad + 2\ \sup\ \ell_1(t)\big|f(t_1, x(\alpha(t_2))) - f(t_2, x(\alpha(t_2)))\big|\\
&\quad + 2\ \sup\\
&\ell_2(t)\left|\int_0^{\beta(t_1)} g(t_1, s, x(s))\, ds - \int_0^{\beta(t_1)} g(t_2, s, x(\gamma(s)))\, ds\right|\\
&\quad + 2\ \sup\\
&\ell_2(t)\left|\int_0^{\beta(t_1)} g(t_2, s, x(\gamma(s)))\, ds - \int_0^{\beta(t_2)} g(t_2, s, x(\gamma(s)))\, ds\right|\\
&\quad + \omega_r^T(F, \varepsilon)\\
&\leq 2\ \sup\ \frac{\ell_1(t)\, k_1(t)\, |x(\alpha(t_1)) - x(\alpha(t_2))|}{M + |x(\alpha(t_1)) - x(\alpha(t_2))|} + L_1 \omega_r^T(f, \varepsilon)\\
&\quad + 2\ \sup\ \ell_2(t)\left|\int_{\beta(t_2)}^{\beta(t_1)} |g(t_2, s, x(\gamma(s)))|\, ds\right|\\
&\quad + 2\ \sup\ \ell_2(t) \int_0^{\beta_T} |g(t_1, s, x(\gamma(s))) - g(t_2, s, x(\gamma(s)))|\, ds\\
&\quad + \omega_r^T(F, \varepsilon)\\
&\leq \frac{L_1\, K_1 \omega^T(x, \omega^T(\alpha, \varepsilon))}{M + \omega^T(x, \omega^T(\alpha, \varepsilon))} + L_1 \omega_r^T(f, \varepsilon) + L_2 \int_0^{\beta_T} \omega_r^T(g, \varepsilon)\, ds\\
&\quad + L_2 G_r^T \varepsilon + \omega_r^T(F, \varepsilon) \qquad (4.4)
\end{aligned}$$

where,

$$\begin{aligned}
\omega_r^T(f, \varepsilon) &= 2\sup\{|f(t_1, x) - f(t_2, x)| : t_1, t_2 \in [0, T], |t_1 - t_2| \leq \varepsilon\\
&\qquad \text{and } x \in [-r, r]\},\\
\omega_r^T(g, \varepsilon) &= 2\sup\{|g(t_1, s, x) - g(t_2, s, x)| : t_1, t_2 \in [0, T],\\
&\qquad |t_1 - t_2| \leq \varepsilon \text{ and } x \in [-r, r]\},\\
\omega_r^T(F, \varepsilon) &= 2\sup\{|F(t_1, x, y) - F(t_2, x, y)| : t_1, t_2 \in [0, T],\\
&\qquad |t_1 - t_2| \leq \varepsilon, x \in [-(K_1 + f_0), (K_1 + f_0)],\\
&\qquad \text{and } y \in [-W, W]\},
\end{aligned}$$

and

$$G_r^T = 2\sup\{|g(t, s, x)|\ :\ t, s \in [0, T],\ \text{and } x \in [-r, r]\}.$$

From the estimate (4.4) it follows that

$$\begin{aligned}
\omega^T(QX, \varepsilon) &\leq \frac{L_1\, K_1 \omega^T(X, \varepsilon)}{M + \omega^T(X, \varepsilon)} + L_1 \omega_r^T(f, \varepsilon)\\
&\quad + L_2 \int_0^T \omega_r^T(g, \varepsilon)\, ds + L_2 G_r^T \varepsilon. \qquad (4.5)
\end{aligned}$$

Since the function $f$ is continuous on $[0, T] \times [-r, r]$ and $g$ is continuous on $[0, T] \times [0, T] \times [-r, r]$, they are continuous there and therefore, we have that $\omega^T(\alpha, \varepsilon) \to 0$, $\omega_r^T(f, \varepsilon) \to 0$ and $\omega_r^T(g, \varepsilon) \to 0$ as $\varepsilon \to 0$. Therefore, from (4.5) it follows that

$$\omega_0^T(QX) \leq \frac{L_1\, K_1 \omega_0^T(X)}{M + \omega_0^T(X)}$$

which further implies that

$$\omega_0(QX) \leq \frac{L_1\, K_1 \omega_0(X)}{M + \omega_0(X)}. \qquad (4.6)$$

Now, let $X$ be a non-empty subset of $\overline{\mathcal{B}}_r(0)$. Then for any $x, y \in X$ and $t \in \mathbb{R}_+$,

$$G(Qx, Qy, Qy)$$

$$
\begin{aligned}
&= 2\ sup\ |Qx(t) - Qy(t)| \\
&\leq 2\ sup\ \ell_1(t)|f(t, x(\alpha(t))) - f(t, y(\alpha(t)))| \\
&\quad + 2\ sup\ \ell_2(t) \int_0^{\beta(t)} |g(t, s, x(\gamma(s)))|\, ds \\
&+ 2\ sup\ \ell_2(t) \int_0^{\beta(t)} |g(t, s, y(s))|\, ds \\
&\leq 2\ sup\ \frac{\ell_1(t)k_1(t)|x(\alpha(t)) - y(\alpha(t))|}{M + |x(\alpha(t)) - y(\alpha(t))|} \\
&+ 2\ sup\ \ell_2(t) \int_0^{\beta(t)} |g(t, s, x(\gamma(s)))|\, ds \\
&\quad + 2\ sup\ \ell_2(t) \int_0^{\beta(t)} |g(t, s, y(s))|\, ds \\
&\leq \frac{L_1 K_1 k_1(t)\ 2\ sup\ |x(\alpha(t)) - y(\alpha(t))|}{M + |x(\alpha(t)) - y(\alpha(t))|} \\
&+ 4L_2 \int_0^{\beta(t)} b(t, s)\, ds \qquad (4.7)
\end{aligned}
$$

Hence,

$$\operatorname{diam} QX(t) \leq \frac{L_1 K_1 \operatorname{diam} X(\alpha(t))}{M + \operatorname{diam} X(\alpha(t))} + 4L_2 w(t).$$

Taking the limit superior as $t \to \infty$ in the above inequality yields

$$
\begin{aligned}
\limsup_{t\to\infty} \operatorname{diam} QX(t) &\leq \frac{L_1 K_1 \limsup_{t\to\infty} \operatorname{diam} X(\alpha(t))}{M + \limsup_{t\to\infty} \operatorname{diam} X(\alpha(t))} \\
&\leq \frac{L_1 K_1 \limsup_{t\to\infty} \operatorname{diam} X(t)}{M + \limsup_{t\to\infty} \operatorname{diam} X(t)}. \qquad (4.8)
\end{aligned}
$$

Further, using the measure of noncompactness $\mu_a$ defined by the formula (3.8) and keeping in mind the estimates (4.6) and (4.8), we obtain

$$
\begin{aligned}
\mu_a(QX) &= \max\left\{\omega_0(QX)\,,\ \limsup_{t\to\infty}\ \operatorname{diam} QX(t)\right\} \\
&\leq \max\left\{\frac{L_1\, K_1 \omega_0^T(X)}{M + \omega_0^T(X)}\,,\ \frac{L_1 K_1 \limsup_{t\to\infty} \operatorname{diam} X(t)}{M + \limsup_{t\to\infty} \operatorname{diam} X(t)}\right\} \\
&\leq \frac{L_1\, K_1 \max\left\{\omega_0^T(X)\,,\ \limsup_{t\to\infty} \operatorname{diam} X(t)\right\}}{M + \max\left\{\omega_0^T(X)\,,\ \limsup_{t\to\infty} \operatorname{diam} X(t)\right\}} \\
&= \frac{L_1 K_1 \mu_a(X)}{M + \mu_a(X)}. \qquad (4.9)
\end{aligned}
$$

From the above estimate we infer that $\mu_a(QX) \leq \psi(\mu_a(X))$, where $\psi(r) = \frac{L_1 K_1 r}{M+r}$. Hence we apply corollary 2.4 to deduce that the operator $Q$ has a fixed point $x$ in the ball $\overline{\mathcal{B}}_r(0)$. Obviously $x$ is a solution of the FIE (3.1). Moreover, taking into account that the image of the space $E$ under the operator $Q$ is contained in the ball $\overline{\mathcal{B}}_r(0)$ we infer that the set $\mathrm{Fix}(Q)$ of all fixed points of $Q$ in $E$ is contained in $\overline{\mathcal{B}}_r(0)$. Obviously, the set $\mathrm{Fix}(Q)$ contains all solutions of the FIE (3.1). On the other hand, from Remark , we conclude that the $\mathrm{Fix}(Q)$ belongs to the family ker $\mu_a$. Now, taking into account the description of sets belonging to ker $\mu_a$ (given in Section 2) we deduce that all solutions for the FIE (3.1) 'are globally uniformly attractive on $\mathbb{R}_+$. This completes the proof.

Next, we prove the global asymptotic attractivity results for the FIE (3.1). We need the following hypotheses in the sequel.

($B_1$) $F(t, 0, 0) = 0$ for all $t \in \mathbb{R}_+$, and

($B_2$) $\lim_{t\to\infty} k_1(t) = 0$, where the function $k_1$ is defined as in hypothesis ($A_1$).

**Theorem 4.2.** Assume that the hypotheses (($A_0$)-($A_6$) and ($B_1$)-($B_2$) hold. Then the FIE (3.1) has at least one solution in the space $BC(\mathbb{R}_+, \mathbb{R})$. Moreover, solutions are uniformly globally asymptotically attractive to zero solution on $\mathbb{R}_+$.

**proof.** Consider the closed ball $\overline{\mathcal{B}}_r(0)$ in the Banach space $BC(\mathbb{R}_+, \mathbb{R})$, where the real number $r$ is given as in the proof of Theorem 4.1 and define a mapping $Q : \overline{\mathcal{B}}_r(0) \to \overline{\mathcal{B}}_r(0)$ by (4.1). Then $Q$ maps the closed ball $\overline{\mathcal{B}}_r(0)$ into $\overline{\mathcal{B}}_r(0)$ such that $Qx(t) \leq Q(Qx(t))$. We show that $Q$ is a nonlinear $\mathcal{M}$-set-contraction with respect to the measure $\mu_c$ of noncompatness in $BC(\mathbb{R}_+, \mathbb{R})$. Let $x \in \overline{\mathcal{B}}_r(0)$ be arbitrary. Then we have

$$
\begin{aligned}
|Qx(t)| &\leq \left| F\Big(t, f(t, x(\alpha(t))), \int_0^{\beta(t)} g(t, s, x(\gamma(s)))\, ds\Big) \right| \\
&\leq \frac{\ell_1(t)k_1(t)\,|x(\alpha(t))|}{M + |x(\alpha(t))|} + \ell_2(t)w(t) \\
&\leq \frac{\ell_1(t)k_1(t)\,\|x\|}{M + \|x\|} + \ell_2(t)w(t) \\
&\leq L_1 k_1(t)\ + L_2 w(t) \qquad (4.10)
\end{aligned}
$$

for all $t \in \mathbb{R}_+$. This further implies that

$$\|QX(t)\| \leq L_1\, k_1(t) + L_2 w(t).$$

Taking the limit superior in the above inequality, we obtain

$$\limsup_{t\to\infty} \|QX(t)\| \leq L_1\ \limsup_{t\to\infty}\ k_1(t) + L_2 \limsup_{t\to\infty}\ w(t) = 0. \qquad (4.11)$$

Further, using the measure of noncompactness $\mu_b$ defined by the formula (3.10) and keeping in mind the estimates (4.6) and (4.11), we obtain

$$
\begin{aligned}
\mu_b(QX) &= \max\left\{\omega_0(QX)\,,\ \limsup_{t\to\infty}\ \|QX(t)\|\right\} \\
&\leq \max\left\{\frac{L_1 K_1 \omega_0(X)}{M + \omega_0(X)}\,,\ 0\right\} \\
&\leq L_1 K_1 \max\left\{\frac{\omega_0(X)}{M + \omega_0(X)}\,,\ 0\right\} \\
&= \frac{L_1 K_1 \mu_b(X)}{M + \mu_b(X)}. \qquad (4.12)
\end{aligned}
$$

Since $\frac{L_1 K_1 r}{M+r} < r$ for $r > 0$, from the above estimate we infer that $Q$ is a $\mathcal{M}$-set-contraction on $\overline{\mathcal{B}}_r(0)$ with respect to the measure of noncompactness $\mu_b$. Hence we apply corollary 2.4 to deduce that the operator $Q$ has a fixed point $x$ in the ball $\overline{\mathcal{B}}_r(0)$. Obviously $x$ is a solution of the FIE (3.1). Moreover, taking into account that the image of $\overline{\mathcal{B}}_r(0)$ under the operator $Q$ is contained in the ball $\overline{\mathcal{B}}_r(0)$ we infer that the set $\mathrm{Fix}(Q)$ of all fixed points of $Q$ is contained in $\overline{\mathcal{B}}_r(0)$. Obviously, the set $\mathrm{Fix}(Q)$ contains all solutions of the equation (3.1). On the other hand, from Remark 3.1 we conclude that the set $\mathrm{Fix}(Q)$ belongs to the family $\ker\mu_b$. Now, taking into account the description of sets belonging to $\ker\mu_b$ (given in Section 3) we deduce that all solutions of the equation (3.1) are uniformly globally asymptotically attractive to the zero solution on $\mathbb{R}_+$.

Finally, while concluding this paper, we mention that our global attractivity results include several local as well as global existence and attractivity results for different types of the nonlinear integral and quadratic integral equations including those considered in Kuczma [20], Banas Rzepka [7], Hu and Yan [19], and several others mentioned therein as the special cases.

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